ON FUNDAMENTAL FUNCTIONS OF LAGRANGEAN INTERPOLATION

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Introduction. It was shown in a number of recent papers that interpolation with fundamental abscissas chosen at the roots of various orthogonal polynomials is of considerable interest.

Let $I=[\alpha, \beta]$ be a closed interval on the real axis, and let $p(x)$ be greater than zero in I. The orthogonal polynomials with respect to $p(x)$ will be denoted by $\phi_n(x)$. Thus, by definition,

$$\int_\alpha^\beta p(x)\phi_n(x)\phi_m(x)dx = \delta_{nm}, \qquad n, m = 0, 1, 2, \cdots .$$

Erdös and Turán* proved important properties of the interpolating polynomial for the case when the zeros of $\phi_n(x)$ are taken for abscissas of interpolation. The theorems so deduced enabled the authors cited to draw important conclusions concerning the distribution of the roots of orthogonal polynomials. The proofs of Erdös and Turán are based on some properties of the fundamental functions of interpolation

$$l_k^n(x) = \frac{\phi_n(x)}{(x - x_k^n)\phi_n'(x_k^n)},\dagger$$

which we do not intend to repeat here. However, it is our purpose to add a few theorems concerning the properties of these fundamental functions. Our main result is the following theorem:

THEOREM. *If $M \geqq p(x) \geqq m > 0$, and if $p(x)$ is continuous in the finite interval $I=[\alpha, \beta]$ and the abscissas of interpolation are chosen as described above, then the maximum of $l_k^n(x)$ in $[\alpha+\epsilon, \beta-\epsilon]$ tends to one as n tends to infinity for all k for which $\alpha+\epsilon \leqq x_k^n \leqq \beta-\epsilon$, ϵ being an arbitrary positive number.*

We shall make use of the following relations:

$$(1) \qquad \int_\alpha^\beta p(x) l_i^n(x) l_k^n(x)dx = 0, \quad \text{if} \quad i \neq k,$$

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* *On interpolation.* I, Annals of Mathematics, (2), vol. 38 (1937), pp. 142–155.

† The function $l_k^n(x)$ is not the nth power of $l_k(x)$; it stands for $l_k^{(n)}(x)$. Similarly, x_k^n stands for $x_k^{(n)}$. The upper indices will be omitted completely in all formulas where there is no risk of misunderstanding.