A GENERALIZATION OF A PROPERTY OF HARMONIC FUNCTIONS*

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1. Introduction. A well known theorem of Bôcher and Koebe characterizes a function $u(x, y)$ as harmonic in a region A if u is of class C' in A † and if the integral of its normal derivative is zero around every circle C in A. This theorem has been generalized by Gergen‡ as follows:

If $v(x, y)$ is harmonic and positive in A, if $u(x, y)$ is of class C' in A, and if

$$(1)\qquad \int_C v\frac{\partial u}{\partial n}ds = \int_C u\frac{\partial v}{\partial n}ds$$

for every circle C interior to A, then u is harmonic in A.

The Bôcher-Koebe result is secured from Gergen's theorem by choosing $v(x, y)\equiv 1$ in A. Gergen's theorem in turn is a special case of the following theorem concerning a general linear partial differential equation of the second order which is self-adjoint and of elliptic type:

THEOREM 1. *Consider the differential expression*

$$(2)\qquad L(z) = az_{xx} + 2bz_{xy} + cz_{yy} + dz_x + ez_y + fz$$

whose coefficients $a, b, \cdots, f$ are functions of (x, y) of class C'' in A which satisfy the conditions

$$(3)\qquad a_x + b_y = d,\qquad b_x + c_y = e,\qquad ac - b^2 > 0$$

in A. Let

$$(4)\qquad \lambda(z) = az_x + bz_y + dz,\qquad \mu(z) = bz_x + cz_y + ez.$$

Let $v(x, y)$ be a function of class C'' which never vanishes in A and which satisfies $L(v)=0$. If $u(x, y)$ is of class C' in A, and if

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† A function $u(x, y)$ is of class $C^{(n)}$ in A if it is continuous and has continuous partial derivatives in A of all orders up to and including the nth.

‡ J. J. Gergen, *Note on a theorem of Bôcher and Koebe*, this Bulletin, vol. 37 (1931), pp. 591–596. See also S. Saks, *Note on defining properties of harmonic functions*, this Bulletin, vol. 38 (1932), pp. 380–382.