FORMAL SYNTHESIS OF TWO PERIODIC CORRESPONDENCES, OF PERIOD FIVE AND SEVEN, RESPECTIVELY*

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Since the illumination of binary doubly quadratic forms by the theory of elliptic functions, it has been generally the custom to study them in no other light. Even Poncelet's picture of two circles, or a pencil of circles, receives scant attention from the general student. This situation resembles a monopoly and is adverse to the progress of diversified industries. A (2, 2) correspondence between variables (x) and (y) creates sequences of values, each of which is considered first as a y, then as an x. I shall consider one such relation, one which is symmetric and periodic. Its period shall be five.

Denote by $\Phi=0$, or $\Phi(x, y)=\Phi(y, x)=0$, a (2, 2) correspondence, symmetric in x and y. For convenience, we may regard it sometimes as homogeneous in x_1 and x_2, sometimes in y_1 and y_2, or at will as non-homogeneous.

Being symmetric, it has a unique representation quadratic in the two combinations $(x+y)$ and xy, or homogeneous in the three, x_1y_1, $(x_1y_2+x_2y_1)$, x_2y_2. If these combinations be taken as line-coordinates while x, y are point parameters on a conic, the corresponding equation $\Phi=0$ becomes the line-equation of a second conic, and we have Poncelet's picture.

Let five quantities, p, q, r, s, t, constitute a closed series or cycle, each one with its successor forming a pair x, y which satisfies the equation $\Phi=0$. As that symmetric equation has only five constants, it is completely determined by these five pairs. Hence we obtain

$$(1)\qquad F(x, y)=\begin{vmatrix} p^2q^2 & pq(p+q) & (p+q)^2 & pq & p+q & 1 \\ q^2r^2 & qr(q+r) & \cdot & \cdot & \cdot & 1 \\ r^2s^2 & \cdot & \cdot & \cdot & \cdot & 1 \\ s^2t^2 & \cdot & \cdot & \cdot & \cdot & 1 \\ t^2p^2 & \cdot & \cdot & \cdot & \cdot & 1 \\ x^2y^2 & xy(x+y) & (x+y)^2 & xy & x+y & 1 \end{vmatrix}=0.$$

This equation includes extraneous factors, $p-r$, $q-s$, $r-t$, $s-p$, $t-q$. To reduce it by exclusion of these factors and express $\Phi(x, y)$

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