NOTE ON AN ELEMENTARY PROBLEM OF INTERPOLATION

P. ERDÖS AND G. GRÜNWALD

The unique polynomial of degree $n-1$ assuming the values $y_1, y_2, \cdots, y_n$ at the abscissas $x_1, x_2, \cdots, x_n$, respectively, is given by the Lagrange interpolation formula

$$(1) \qquad L_n(x) = y_1 l_1(x) + y_2 l_2(x) + \cdots + y_n l_n(x),$$

where

$$(2) \qquad l_k(x) = \frac{\omega(x)}{\omega'(x_k)(x - x_k)}, \qquad k = 1, 2, \cdots, n,$$

(fundamental polynomials of the Lagrange interpolation) and the polynomial $\omega(x)$ is defined by

$$(3) \qquad \omega(x) = c(x - x_1)(x - x_2) \cdots (x - x_n),$$

where c denotes an arbitrary constant not equal to zero.

In this note we prove the following theorem:

THEOREM. *In the Lagrange interpolation formula let* $x_k = x_k^{(n)} = \cos (2k-1)\pi/2n = \cos \theta_k^{(n)}$, $(k=1, 2, \cdots, n)$, *which implies* $\omega(x) = T_n(x) = \cos (n \arc \cos x) = \cos n\theta$ (*Tschebycheff polynomial*). *Then*

$$(4) \qquad | l_k^{(n)}(x) | = \left| \frac{\omega(x)}{\omega'(x_k)(x - x_k)} \right| < \frac{4}{\pi}, \qquad -1 \leqq x \leqq +1,$$

for all n *and* k, *and furthermore*

$$(5) \qquad \lim_{n\to\infty} | l_1^{(n)}(+1) | = \lim_{n\to\infty} | l_n^{(n)}(-1) | = \frac{4}{\pi} \cdot$$

In this connection Fejér* proved for all n, k, and x, $(-1 \leqq x \leqq +1)$,

$$(6) \qquad | l_k^{(n)}(x) | < 2^{1/2}.$$

Of course (5) implies that inequality (4) is the best possible in the following sense: For any $\epsilon > 0$ there exist values of n, k, and x,

* L. Fejér, *Lagrangesche Interpolation und die zugehörigen konjugierten Punkte*, Mathematische Annalen, vol. 106 (1932), pp. 1–55; see pp. 10, 11. This paper will hereafter be referred to as L.