A NOTE ON NUMBERS OF THE FORM $a^2+\alpha b^2+\beta c^2+\alpha\beta d^2$

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It has long been known that the only sets of values of α, β which give rise to universal forms $a^2+\alpha b^2+\beta c^2+\alpha\beta d^2$ are $\alpha=1$, $\beta=1, 2, 3$; $\alpha=2$, $\beta=2, 3, 4, 5$. We give here a theorem from which the universal character of all these multiplicative universal forms can be readily established. The methods and arguments used are completely algebraic in character. This note is closely allied to R. D. Carmichael's paper, *Proof that every positive integer is a sum of four integral squares.** We use the formulas on the first page of his proof without recording them here.

We remark that every prime number p is a divisor of the form $a^2+\alpha b^2+\beta c^2+\alpha\beta d^2$ in which a, b, c, d are relatively prime. For if p is prime to α, a theorem from the theory of quadratic residues states that p is a divisor of $a^2+\alpha b^2+\beta$. If p is not prime to α, take $a=c=d=0$, $b=1$. In both cases the bases are seen to be relatively prime.

We now require that α, β be positive integers with $\beta \geqq \alpha > 0$. The results obtainable also hold for $\alpha=0$, which case has been treated by R. D. Carmichael.†

THEOREM 1. *For $\alpha<3$ and every prime number $p>\beta$ there exists a positive integer $q \leqq [4\beta/(3-\alpha)]^{1/2}$ such that $p\cdot q=a^2+\alpha b^2+\beta c^2+\alpha\beta d^2$.*

For $p<5$ we verify the truth of the theorem directly. Henceforth consider $p\geqq 5$. Since $p>\beta\geqq\alpha$, from the above remark we have a q' such that $pq'=a^2+\alpha b^2+\beta$, where a and b may evidently be taken less than $p/2$. Hence, $pq' \leqq (1+\alpha)p^2/4+\beta$ or $q'\leqq(1+\alpha)p/4+\beta/p < 3p/4+1$; that is, $q'<p$ if $p\geqq 5$, which is the case now under consideration.

Let q be the smallest positive integer such that $pq=a^2+\alpha b^2+\beta c^2+\alpha\beta d^2$. From the above we have $q<p$. Also a, b, c, d are relatively prime, for otherwise the square of their greatest common divisor would divide q, leaving pq_1 in the same form as pq with $q_1<q$ contrary to the hypothesis that q has the smallest value possible.

We now consider the forms in the parentheses of the formula that would correspond to Carmichael's (2):

* Duke Mathematical Journal, vol. 2 (1936), pp. 243–245.

† American Mathematical Monthly, vol. 44 (1937), pp. 81–86.