EXTENSIONS OF FUNCTIONALS ON COMPLEX LINEAR SPACES

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The theorem of Hahn-Banach* states that any linear (continuous) functional defined on a linear subspace of a normed linear space can be extended to the whole space without increasing the norm or modulus of the functional. This theorem however deals only with real linear spaces and real-valued functionals. It is unfortunately not entirely possible to remove the restriction to real values and real coefficients. The present paper discusses this question and investigates the extent to which the theorem of Hahn-Banach remains valid in the general case of complex linear spaces. It is of interest to notice that the difficulties which appear are not due to the possibility that a linear space may be of infinite dimension; the difficulties are present essentially even in the finite dimensional case.

To simplify the exposition let us agree to call a functional $f(x)$ a *real linear* functional if, for any real numbers a, b and any two elements x, y of a linear space L, the relation

$$f(ax + by) = af(x) + bf(y)$$

holds. If the space L is complex linear, and if the above relation remains valid even for all pairs of complex numbers a, b, then the functional will be called *complex linear*. On a complex linear space we may of course have a real linear functional which is not complex linear. Let us remark that by a real subspace of a complex linear space shall be meant any subspace which contains all real linear combinations of any finite number of its elements. Such a subspace may also contain some or all complex linear combinations of its elements. A functional defined over a real subspace will be called *complex linear* if the above relation holds for all complex linear combinations which do not lead to elements outside the subspace. We are of course concerned only with the extension of complex linear functionals.

The theorem of Hahn-Banach has this analogue:

THEOREM 1. *Let l be any complex linear subspace of a normed complex linear space L. Let $f(x)$ be any complex linear functional defined on l, having a norm M. Then there always exists a complex linear functional $F(x)$ defined on L, which coincides with $f(x)$ in l, and which has the same norm M on L.*

* See Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932, p. 28 and p. 55.