NOTE ON DEDUCED PROBABILITY DISTRIBUTIONS

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In this Bulletin, December, 1936, A. H. Copeland* resumed the study of the problem first suggested by H. Poincaré: How can the fact of uniform probability distribution, which we meet so frequently in different games of chance, be explained? Recently E. Hopf devoted a profound essay† to this question and he has just published a short note‡ dealing with his principal results. I want to contribute a quite simple remark which seems to show how far the results are independent of the particular form of dynamical equations.

We assume that there exists a density function $f(x)$ for the one-dimensional variable x, such that $\int_a^b f(x)dx$ denotes the probability that the value of x falls in the interval (a, b) and $\int_{-\infty}^{\infty} f(x)dx=1$. If between x and y there is established a one-to-one correspondence

$$(1) \qquad y = y(x), \qquad x = x(y),$$

the given density function $f(x)$ leads to a new density function $g(y)$ defined by

$$(2) \qquad g(y) = f(x)\frac{dx}{dy}\cdot$$

The integral $\int_a^b g(y)dy$ gives, of course, the probability that y belongs to the interval (a, b) and $\int_{-\infty}^{\infty} g(y)dy=1$.

Now we suppose y to be an "angular" variable, that is, instead of y we consider the new variable:

$$(3) \qquad \eta = y - [y], \qquad 0 \leqq \eta < 1,$$

and try to determine the probability distribution $\phi(\eta)$ of η. Evidently, if ν is a positive or negative integer, the probability density of η is given by

$$(4) \qquad \phi(\eta) = \sum_\nu g(\eta+\nu) = \sum_\nu f(x_\nu)\left(\frac{dx}{dy}\right)_{x=x_\nu}; \qquad x_\nu = x(\eta+\nu).$$

* Vol. 42, p. 895.

† Journal of Mathematics and Physics, Massachusetts Institute of Technology, vol. 13 (1934).

‡ Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 46 (1936), p. 179.