THE CALCULUS OF VARIATIONS APPLIED TO NÖRLUND'S SUM*

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Nörlund's sum† function

$$\mathop{\mathbf{S}}_c^u f(x)\Delta x$$

has many resemblances to a definite integral. The purpose of the present note is to point out how some of the classical methods of the calculus of variations can be applied to such a sum. It may be that the field will prove fruitful for further research.

We shall consider the problem of minimizing (maximizing) the sum

$$\mathop{\mathbf{S}}_c^b F(x, y, \Delta y, \Delta^2 y, \cdots, \Delta^n y)\Delta x, \tag{1}$$

where we have exactly the same understanding of what constitutes a minimum as in the classical problem of the definite integral.

1. *Euler's Equation.* We shall seek a necessary condition that a continuous real y minimize

$$\mathop{\mathbf{S}}_c^b F(x, y, \Delta y)\Delta x \tag{2}$$

similar to Euler's equation for the corresponding integral. The condition of fixed end points in the integral problem is here replaced by the condition that y be fixed over the interval $c \leqq x \leqq c+1$ and at the point b.

For brevity in writing denote Δy by y', $F(x, y, \Delta y)$ by $F(x)$, and assume that y is continuous and that F, F_y, and $F_{y'}$ are continuous in their arguments throughout all neighborhoods considered in the sequel.

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† Milne-Thompson, *The Calculus of Finite Differences*, page 201. In the present paper the difference interval is assumed to be 1.