NOTE ON AN ELEMENTARY GEOMETRIC EXISTENCE THEOREM

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1. *Introduction.* By the $\prod$-moment of a set of points $c_1, c_2, \cdots, c_m$, $(m>1)$, with respect to a particular c_q we mean the product of the $m-1$ distances from c_q to the other c's. The object of the present note is to prove the following proposition:

THEOREM 1. *For any given set of distinct points $a_1, a_2, \cdots, a_n$, $(n>1)$, in the plane, there will exist a set of points $b_1, b_2, \cdots, b_n$, distinct from the a's, such that the $\prod$-moment of the a's and b's with respect to a_i is the same for all i.*

This theorem is obviously a special case of the following more general theorem:

THEOREM 2. *Let $a_1, a_2, \cdots, a_n$, $(n>1)$, be a set of distinct points in a plane and let $d_{i,j}$ be the distance from a_i to a_j, $(i\neq j)$.* Let r_i be any given ray (half line) issuing from a_i. Then for every sufficiently small positive number δ, say for $\delta \leqq \delta_0$, there exists a set of points $b_1, b_2, \cdots, b_n$ satisfying the two conditions* (a) *the point b_i is on the ray r_i, and* (b) *if $D_{i,j}$ is the distance from a_i to b_j, then*

$$(1) \qquad d_{i,1}d_{i,2}\cdots d_{i,i-1}d_{i,i+1}\cdots d_{i,n}D_{i,1}D_{i,2}\cdots D_{i,n} = \delta$$

for all values of i.

2. *An Outline of the Procedure.* An algebraic proof of the existence of the desired solutions of (1) seems to be difficult, so we prefer to use the method of successive approximations. The formal details of our procedure are a little tedious, but the underlying idea is relatively simple, namely: Since δ is small, the point b_i will be much closer to a_i then to any of the other a's or b's. Thus, if b_i is shifted slightly, the quantity $D_{i,i}$ will change relatively much more than any of the other distances. This has the effect of making the product, $\prod$, in (1) approximately a

* Throughout this note, i and j will range over the values $1, 2, 3, \cdots, n$, and k over the values $1, 2, 3, \cdots$.