FUNCTIONS OF COPRIME DIVISORS OF INTEGERS

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1. *Unique Decompositions.* If a set U of distinct positive integers $1, u_1, u_2, \cdots$ is such that*

$$(1) \qquad (u_i, u_j) = 1, \qquad i \neq j, \qquad i, j = 1, 2, \cdots ,$$

we call U a *coprime set.* If to U we adjoin all positive integral powers $u_1^{\alpha_1}, u_2^{\alpha_2}, \cdots, \alpha_1>0, \alpha_2>0, \cdots$ of integers in U, we get the *extended set* $E(U)$. If m is in $E(U)$, we call m a *U-integer.*

THEOREM 1. *If $n>1$ is representable as a product of powers of integers >1 in U, the representation is unique (up to permutations of the factors), say*

$$(2) \qquad n = u_1^{c_1} \cdots u_r^{c_r}, \qquad u_i > 1, \qquad c_i > 0, \qquad i = 1, \cdots, r.$$

For, by the definition of U, the u_i in (2) are distinct, and by (1) a prime p such that $p|n$ is such that $p|u_j$ for precisely one j, $0<j\leqq r$. We call (2) the *U-decomposition* of n.

Obviously there exist U's such that some $n>1$ are not U-decomposable. From the fundamental theorem of arithmetic we have the following theorem:

THEOREM 2. *If $P\equiv p_1, p_2, \cdots$ is the set of all positive primes, the only U such that every integer $n>1$ is U-decomposable is $U\equiv P$.*

We shall consider also another type of unique decomposition, valid for all $n>1$, which has the distinguishing property of U-decomposition as in (2), namely, *every $n>1$ is uniquely a product of powers of coprime integers >1.*

If the integer $s>0$ is divisible by the square of no prime, we call s simple. Let $S\equiv 1, s_1, s_2, \cdots$ be the set of all distinct simple integers; S includes P and is not a coprime set. Without confusion we may denote by $E(S)$ the set obtained by adjoining to S all positive integral powers $s_1^{\alpha_1}, s_2^{\alpha_2}, \cdots, \alpha_1>0, \alpha_2>0, \cdots$, of simple integers.

Let $n=p_1^{a_1}\cdots p_r^{a_r}$ be the P-decomposition of n. If $a_1, \cdots, a_r$ are all different, this is by definition also the *S-decomposition.* If

* In the customary notations, (m, n) is the G.C.D. of m, n, and $m|n$ signifies that m divides n arithmetically.