THE SUCCESSIVE ITERATES OF THE STIELTJES KERNEL EXPRESSED IN TERMS OF THE ELEMENTARY FUNCTIONS*

BY D. V. WIDDER

1. *Introduction.* The Stieltjes kernel is the function

$$H_1(\xi, \eta) = \frac{1}{\xi + \eta} \cdot$$

We define its successive iterates by the recurrence relation

$$H_n(\xi, \eta) = \int_0^\infty \frac{H_{n-1}(t, \eta)dt}{\xi + t}, \qquad (n = 2, 3, 4, \cdots).$$

That these integrals all exist will appear from later considerations. Simple computation shows that

$$H_2(\xi, \eta) = \int_0^\infty \frac{dt}{(\xi + t)(t + \eta)} = \frac{\log \xi - \log \eta}{\xi - \eta}, \qquad (\xi > 0, \eta > 0).$$

It is natural to inquire if it is also possible to express the higher iterates in terms of the elementary functions. It is the purpose of the present note to prove that this is the case. We show, in fact, that $H_n(\xi, \eta)$ is a linear combination of the functions

$$\frac{(\log \xi - \log \eta)^{2k+1}}{(2k+1)!(\xi - \eta)}, \qquad \frac{(\log \xi - \log \eta)^{2k}}{(2k)!(\xi + \eta)}, \qquad (k = 0, 1, 2, \cdots),$$

the constants of combination being the coefficients of the power series expansion of $(\pi/\sin \pi s)^n$. The precise result to be proved is contained in Theorem 2 of this paper, stated as follows:

THEOREM 2. *If* $0<\xi<\infty$, $0<\eta<\infty$, *then*

$$H_{2n}(\xi, \eta) = \sum_{k=1}^{n} \frac{A_{2n,2k}}{(2k-1)!} \frac{[\log \xi - \log \eta]^{2k-1}}{\xi - \eta}, \quad (n = 1, 2, \cdots),$$

$$H_{2n+1}(\xi, \eta) = \sum_{k=0}^{n} \frac{A_{2n+1,2k+1}}{(2k)!} \frac{[\log \xi - \log \eta]^{2k}}{\xi + \eta}, \qquad (n = 0, 1, 2, \cdots),$$

* Presented to the Society, September 10, 1937.