SOME THEOREMS ON TENSOR DIFFERENTIAL INVARIANTS

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1. *Introduction.* In the theory of algebraic invariants there is a theorem which states that if an absolute invariant be written as the quotient of two relatively prime polynomials, then the numerator and denominator are relative invariants.* If we consider absolute scalar differential invariants of a metric (or affine) space, then it is possible to prove a similar theorem regarding them. In the course of the proof we give a new proof of the fact that in a relation of the form (2) the ϕ must be a power of the Jacobian of the coordinate transformation. (In the algebraic theory the u^i_j are of course constants.) This proof involves the use of the differential equations satisfied by the scalar.† In this proof it is not necessary to restrict B and ϕ to be polynomials in their arguments as is done in the usual proof of the corresponding theorem in the invariant theory. It is sufficient to assume that ϕ possesses first derivatives with respect to the u^i_j and that $B(\bar{g})$ is an analytic function of ϵ in the neighborhood of $\epsilon=0$. We also extend the theorem to the case of tensor differential invariants of the form (5).

2. *Scalar Differential Invariants.* We consider the differential invariants of a metric space V_n with a quadratic form $g_{ij}dx^idx^j$. Let

$$A\left(g_{ij};\ \frac{\partial g_{ij}}{\partial x^k};\ \cdots;\ \frac{\partial^p g_{ij}}{\partial x^k\cdots\partial x^l}\right)$$

be an absolute scalar invariant of V_n which we take to be rational in its arguments. We can then write A in terms of the g_{ij} and their extensions $g_{ij,k\cdots l}$, and we have

$$A(g_{ij};\ 0;\ g_{ij,kl};\ \cdots) = \frac{B(g_{ij};\ 0;\ g_{ij,kl};\ \cdots)}{C(g_{ij};\ 0;\ g_{ij,kl};\ \cdots)},$$

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* See, for example, H. W. Turnbull, *The Theory of Determinants, Matrices, and Invariants*, p. 277.

† T. Y. Thomas and A. D. Michal, *Differential invariants of relative quadratic differential forms*, Annals of Mathematics, vol. 28 (1927), p. 679.