ON THE SINGULARITIES OF AN ANALYTIC FUNCTION*

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1. *Introduction*. We shall consider an analytic function $f(z)$ represented by the series $\sum_{n=0}^{\infty} a_n z^n$ whose circle of convergence we shall suppose for simplicity to be the *unit* circle with center at the origin of the complex plane. Our purpose is to give simple generalizations of certain theorems of Pringsheim and Hadamard relative to the singularities of $f(z)$ on C, the circumference of the circle of convergence.

With the present hypotheses and notation the theorems in question may be formulated as follows:

THEOREM OF PRINGSHEIM.† *In order that $z=1$ be a simple pole of $f(z)$ and that there be no further singularity of $f(z)$ on C it is necessary and sufficient that*

$$\overline{\lim_{n\to\infty}} \left| a_{n+1} - a_n \right|^{1/n} < 1.$$

THEOREM OF HADAMARD.‡ *In order that there be just one simple pole and no further singularity of $f(z)$ on C it is necessary and sufficient that*

$$\overline{\lim_{n\to\infty}} \left| a_n^2 - a_{n-1}a_{n+1} \right|^{1/n} < 1.$$

2. *Generalizations of the Above Theorems*. We shall first establish a generalization of Pringsheim's theorem.

THEOREM 1. *In order that $z=1$ be a pole of order m of $f(z)$ and that there be no further singularity of $f(z)$ on C it is necessary and sufficient that there exist a polynomial $g(x)$ of degree $m-1$ such that, if we put $A_n = a_n/g(n)$, we have*

$$\overline{\lim_{n\to\infty}} \left| A_{n+1} - A_n \right|^{1/n} < 1.$$

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† A. Pringsheim, *Vorlesungen über Zahlen- und Funktionenlehre*, vol. 2, part 2, 2d ed., 1932, p. 916.

‡ See, for instance, P. Dienes, *The Taylor Series*, p. 333.