SOME THEOREMS ON DOUBLE LIMITS*

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1. *Introduction.* Let $f(x, y)$ be an arbitrary single-valued real function of the real variables x, y defined in the neighborhood of a point $Q(a, b)$, which for simplicity may be taken as $(0, 0)$. The following sufficient (and obviously necessary) condition for the existence of the double limit

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x, y) \tag{1}$$

has been established.

THEOREM 1 (Clarkson).‡ *If $f(x, y)$ has a unique limit as $P(x, y)$ approaches Q on every curve having a tangent at Q, the double limit* (1) *exists.*

The present note is concerned with similar theorems, and for definiteness we state at the outset that the assertion, "$f(P)$ has a limit λ as $P\to Q$ on a point set§ E having Q as a limit point (or $\lim_{P\to Q} f(P) = \lambda$, on E)" shall mean that for each $\epsilon > 0$ there exists a positive $\delta(\epsilon, E)$ such that $|f(P) - \lambda| < \epsilon$ for all points P of E satisfying the condition $0 < |x| + |y| < \delta$.

Theorem 1 naturally suggests a question which is answered by Lemma 1, for convenience in the statement of which we introduce the following definition.

DEFINITION OF PROPERTY L. A class $\{E\}$ of sets E, each having Q as a limit point, will be said to have Property L if and only if any set S whatsoever of points having Q as a limit

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† I gratefully acknowledge my indebtedness to Mr. Hugh J. Hamilton for suggesting Lemma 1, and to Mr. Nelson Dunford for Theorem 5.

‡ Clarkson, *A sufficient condition for the existence of a double limit*, this Bulletin, vol. 38 (1932), pp. 391–392. A theorem essentially the same has been proved by Verčenko and Kolmogoroff, *Über Unstetigkeitspunkte von Funktionen zweier Veränderlichen*, Comptes Rendus, Académie des Sciences, URSS, new series, vol. 1 (1934), pp. 105–107.

§ In particular, on a curve.