THE NUMBER SYSTEM AFTER DEDEKIND

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1. *Introduction.* The rational numbers (positive, negative, and zero) having been defined and the four species for them established, the creation of the irrational numbers is possible in a variety of ways, notably by *regular sequences* and by Dedekind's *cut.*

The first method is, perhaps, easiest for the beginner. The fundamental principle of limits once being accepted in the form of the *definition* of a regular sequence, the detailed developments proceed smoothly, and the system of real numbers is evolved satisfactorily.*

Complete as this method is, there are mathematicians who feel that the definition of irrational numbers by means of the cut is essentially simpler, more elementary, for it creates these numbers before the concept of a limiting process is introduced and thus provides the elements for point sets before any infinite process has been defined.

Addition and subtraction are defined in the simplest manner possible on the basis of the cut. If $A=(a_1, a_2)$ and $B=(b_1, b_2)$ are any two numbers, then $C=A+B$ is the number

$$(c_1, c_2) = (a_1 + b_1, a_2 + b_2).$$

But multiplication cannot be defined in the same way. One would like to define $C=AB$ as the number $(c_1, c_2)=(a_1b_1, a_2b_2)$; but this is impossible since the fractions a_1b_1, a_2b_2 do not yield a cut. True, one can introduce the positive irrationals after the positive fractions, and before the negative numbers and zero, have been defined. The above definition of multiplication as the product of two cuts is now possible. The definition of negative numbers goes through exactly as before, and the system of real numbers is complete. We have lost, however, the earlier definition of a real number as a cut, and this is too high a price to pay for the simplified definition of multiplication.

* Pierpont has given an excellent exposition of this method in his *Functions of Real Variables*, vol. 1, pp. 31–61.