ON A THEOREM OF PLESSNER*

BY NELSON DUNFORD

If $f(t)$ is a real-valued function of a real variable, periodic with period 2π and of bounded variation, then $f(t)$ is absolutely continuous provided $v(u)$, the total variation of $f(t+u)-f(t)$ on any interval of length 2π, tends to zero with u. This theorem, which is the converse of a well known theorem of Lebesgue, has been proved by Plessner, and by Wiener and Young.† Ursell‡ has given some interesting results concerning the total variation of $f(t+u)-f(t)$ for measurable functions $f(t)$, which when combined with the above theorem show (as he has pointed out) that that theorem holds for measurable functions. The papers referred to contain the essential ideas sufficient for a very short proof of the general theorem. In fact, with two additional lemmas which are given below, the proof as given by Plessner holds in the general case.

By an *admissible function* will be meant one which is finite-valued and periodic with period 2π.

THEOREM 1. *If $f(t)$ is admissible and $\lim_{u=0} v(u)=0$, then $v(u)$ is continuous.*

Let u_1, u_2 be any real numbers and $\delta_i=(t_{i-1}, t_i)$ be a partition of the interval $(-\pi, \pi)$. Then, if $t_i' = t_i+u_1$, the intervals $\delta_i' = (t_{i-1}', t_i')$ form a partition of $(-\pi+u_1, \pi+u_1)$ and§

$$\delta_i\{f(t+u_1+u_2)-f(t)\} = \delta_i\{f(t+u_1)-f(t)\} + \delta_i'\{f(t+u_2)-f(t)\},$$

and so $v(u_1+u_2)\leqq v(u_1)+v(u_2)$. This shows that v is finite everywhere; for if $v(u)<K$ on $(-a, a)$, then $v(u)<2K$ on $(-2a, 2a)$.

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† Plessner, *Eine Kennzeichnung der totalstetigen Funktionen*, Journal für Mathematik, vol. 160 (1929), pp. 26–32. Wiener and Young, *The total variation of* $g(x+h)-g(x)$, Transactions of this Society, vol. 35 (1933), pp. 327–340.

‡ Ursell, *On the total variation of* $f(t+\tau)-f(t)$, Proceedings of the London Mathematical Society, (2), vol. 37 (1934), pp. 402–415.

§ If $\delta=(a, b)$, by $\delta f(t+u)$ is meant $f(b+u)-f(a+u)$.