ON AN EXPANSION OF THE REMAINDER IN THE GAUSSIAN QUADRATURE FORMULA*

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1. *Introduction.* The Gaussian quadrature formula

$$\int_0^1 f(x)dx = A_1 f(x_1) + A_2 f(x_2) + \cdots + A_n f(x_n), \tag{1}$$

in which $x_1, x_2, \cdots, x_n$ are roots of Legendre's polynomial

$$P_n(x) = \frac{d^n x^n (x-1)^n}{dx^n},$$

and

$$A_i = \int_0^1 \frac{P_n(x)}{(x - x_i)P_n'(x_i)} dx, \quad (i = 1, 2, \cdots, n),$$

is exact in case $f(x)$ is an arbitrary polynomial of degree not exceeding $2n-1$. Otherwise the formula (1) is only approximate, and the difference between its left and right hand sides represents the error or remainder term which will be denoted in what follows by R_n. The expression of this remainder, obtained, if I am not mistaken, for the first time by A. A. Markoff in 1884, is well known. In this article I shall prove that the remainder in the Gaussian formula can be expanded into a series possessing all the properties of the classical Euler-Maclaurin expansion. This is a noteworthy fact, equally important from the theoretical and from the practical point of view.

2. *Expression of R_n.* In what follows we shall adopt E. Nörlund's definition of the Bernoullian polynomial $B_n(x)$ of order n; and we shall define the periodic function $\overline{B}_n(x)$ by the equations

$$\overline{B}_n(x) = B_n(x), \text{ for } 0 \leqq x < 1;$$

$$\overline{B}_n(x+1) = \overline{B}_n(x), \text{ for all } x.$$

With these notations, we have, for $0 \leqq \theta \leqq 1$,

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