A NOTE ON UNITS IN SUPER-CYCLIC FIELDS

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1. *Comparison of Two Known Results Concerning Cyclotomic Units.* Kummer* first showed that if

$$\zeta = e^{2i\pi/l}$$

with l an odd prime, and if η is a unit in $k(\zeta)$ such that

$$\eta \equiv a \pmod{l},$$

where a is a rational integer, then

$$\eta = \rho^l,$$

where ρ is in $k(\zeta)$, provided none of the Bernoulli numbers

$$(1) \qquad B_1, B_2, \cdots, B_d, \qquad (d = (l-3)/2),$$

is divisible by l. Kummer's proof of this depended on the fact that under the assumptions mentioned there exists an integer c prime to l such that

$$(2) \qquad \eta^c = E_1^{a_1} E_2^{a_2} \cdots E_d^{a_d}.$$

Here

$$E_n = \prod_{i=0}^{d} \epsilon(\zeta^{r^i})^{r^{-2in}},$$

$$\epsilon = \left(\frac{(1-\zeta^r)(1-\zeta^{-r})}{(1-\zeta)\,(1-\zeta^{-1})}\right)^{1/2}.$$

From this we obtain an identity in an indeterminate x by adding a certain multiple of

$$\frac{x^l - 1}{x-1}.$$

Setting $x = e^v$, taking logarithms and differentiating $2n$ times, $(n = 1, 2, \cdots, d)$, we find, using relations in another paper,†

* Journal für Mathematik, vol. 40 (1850), p. 128.

† Transactions of this Society, vol. 31 (1929), pp. 619–620, relations (4) and (5).