ON A THEOREM OF PALEY

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1. *Introduction.* In 1910 I published the following theorem.†

THEOREM 1. *Let $f(x)$ be integrable in the interval $0 \leqq x \leqq 2\pi$ and such that*

$$(1) \qquad |f(x)| \leqq M, \qquad (0 \leqq x \leqq 2\pi),$$

and let the Fourier coefficients a_n, b_n of $f(x)$ satisfy the conditions

$$(2) \qquad |a_n| \leqq \frac{A}{n}, \qquad |b_n| \leqq \frac{B}{n}, \qquad (n = 1, 2, \cdots),$$

where M, A, B are non-negative constants. Then, if $s_n(x)$ denotes the sum of the $(n+1)$ first terms of the Fourier series of $f(x)$, the following inequality holds:

$$(3) \quad |s_n(x)| \leqq M + A + B, \qquad (0 \leqq x \leqq 2\pi, n = 0, 1, 2, \cdots).$$

This criterion of boundedness of the partial sums of the Fourier series of a bounded function follows immediately from the elementary fact that, when $|f(x)| \leqq M$ in $(0, 2\pi)$, then also $|S_n(x)| \leqq M$ in $(0, 2\pi)$, $(n=0, 1, 2, \cdots)$, where $S_n(x)$ is the arithmetic mean of the first $(n+1)$ partial sums of the Fourier series of $f(x)$. Indeed, for an arbitrary infinite series

$$(4) \qquad u_0 + u_1 + u_2 + \cdots + u_n + \cdots,$$

there exists the relation

$$(5) \qquad s_n = S_n + (n+1)^{-1} \sum_{\nu=1}^{n} \nu u_\nu,$$

from which Theorem 1 follows at once.

2. *Paley's Theorem.* Leaving aside various interesting considerations related to Theorem 1 which are known, I shall now pass on to a second boundedness criterion for the partial sums

† *Sur les sommes partielles de la série de Fourier*, Comptes Rendus, vol. 150 (1910), pp. 1299–1302.