NECESSARY AND SUFFICIENT CONDITIONS FOR THE REPRESENTATION OF A FUNCTION BY A DOUBLY INFINITE LAPLACE INTEGRAL*

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1. *Introduction.* By a doubly infinite Laplace integral we mean the improper Stieltjes integral

$$\int_{-\infty}^{\infty} e^{-xt}d\alpha(t).$$

Here $\alpha(t)$ is a function of bounded variation in every finite interval, and

$$\int_{-\infty}^{\infty} e^{-xt}d\alpha(t) = \lim_{R\to\infty}\int_0^R e^{-xt}d\alpha(t) + \lim_{R\to\infty}\int_{-R}^0 e^{-xt}d\alpha(t).$$

We are concerned in this note with the case in which $\alpha(t)$ is an increasing function. We propose to characterize the class of functions which can be represented by such integrals.

In order to foresee what type of result may be expected we first recall certain properties of the moments of a function on a finite interval. Set

$$\mu_n = \int_0^1 y^n d\chi(y), \qquad (n = 0, 1, 2, \cdots), \tag{1}$$

where $\chi(y)$ is a real increasing function. It was proved by F. Hausdorff† that equations (1) have a solution $\chi(y)$ if and only if the sequence $\{\mu_n\}$ is completely monotonic,

$$(-1)^k\Delta^k\mu_n \geqq 0, \quad (n, k = 0, 1, 2, \cdots). \tag{2}$$

If the integer n in (1) is replaced by a continuous variable x, we obtain

$$\mu(x) = \int_0^1 y^x d\chi(y),$$

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† F. Hausdorff, *Momentprobleme für ein endliches Intervall*, Mathematische Zeitschrift, vol. 16 (1923), p. 220.