ALGEBRAIC DIFFERENCE EQUATIONS*

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1. *Introduction.* It was observed by J. C. Charles in 1788† that non-linear algebraic difference equations of the first order, even if algebraically irreducible, might have more than one solution involving an arbitrary periodic function. For instance,

$$y = (x + c)^2, \tag{1}$$

where c is an arbitrary function of period unity, is a solution of

$$[y(x + 1) - y(x)]^2 - 2[y(x + 1) + y(x)] + 1 = 0. \tag{2}$$

The first member of (2) is irreducible, in the domain of rationality of all constants, as a polynomial in $y(x+1)$ and $y(x)$. Still (2) admits, in additon to (1), the solution

$$y = (ce^{\pi i x} + \frac{1}{2})^2, \tag{3}$$

where c is a function of period unity. The complete solution of (2) is given by (1) and (3).

Although the formal aspects of non-linear algebraic difference equations intrigued considerably several of the early French analysts, the general theory of such equations appears to have received, as yet, but scant attention. The first move towards a comprehensive theory seems to be contained in a recent paper by J. L. Doob and myself,‡ in which a definition was given of *irreducible system* of algebraic difference equations, and in which it was shown that every system of such equations is equivalent to a finite set of irreducible systems. The problem now is one of studying irreducible systems. Until an adequate existence theory is developed for systems of non-linear differ-

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† See Biot, Journal de l'Ecole Polytechnique, vol. 4, cahier 11 (1802), p. 182; Poisson, ibid., p. 173; Boole, *Calculus of Finite Differences*, 3d edition, (1880), Chapter 10.

‡ American Journal of Mathematics, vol. 55 (1933), p. 505. (Designated below by α.)