A THEOREM ON FOURIER-STIELTJES INTEGRALS

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1. *Introduction.* If $V(\alpha)$ is a (complex-valued) function of bounded variation in $-\infty<\alpha<\infty$,

$$(1) \qquad \int_{-\infty}^{\infty} |\, dV(\alpha)\,| = M < \infty,$$

then the function

$$(2) \qquad f(x) = \int_{-\infty}^{\infty} e^{ix\alpha} dV(\alpha)$$

is a bounded continuous function in $-\infty<x<\infty$. We denote the class of these functions $f(x)$ by $\mathfrak{B}$. The *distribution function* $V(\alpha)$ which generates $f(x)$ is essentially unique* and we shall call the number M the *norm* of $f(x)$.

A sub-class of $\mathfrak{B}$ is the class $\mathfrak{B}$ of those functions $f(x)$ whose distribution function is (real and) non-decreasing. The class of the latter functions coincides with the class of the so-called *positive-definite* functions for which I have recently given an independent characterization.† It is immediately seen that the functions of $\mathfrak{B}$ consist of all expressions

$$f_1(x) - f_2(x) + if_3(x) - if_4(x),$$

in which f_1, f_2, f_3, f_4 are any positive-definite functions. This indirect characterization of the class $\mathfrak{B}$ is of no interest. But we shall describe the class $\mathfrak{B}$ by an entirely different *direct* property, which is an imitation of a well known criterion due to F. Riesz.

THEOREM. *In order that a bounded continuous function $f(x)$ be a function of $\mathfrak{B}$ with norm $\leqq M$ it is necessary and sufficient*

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* See S. Bochner, *Vorlesungen über Fouriersche Integrale*, Leipzig, 1932, p. 18 ff.

† Loc. cit. Compare also F. Riesz, *Über Sätze von Stone und Bochner*, Acta Szeged, vol. 6 (1933), pp. 184–198; and, for the case of several variables, S. Bochner, *Monotone Funktionen, Stieltjessche Integrale und harmonische Analyse*, Mathematische Annalen, vol. 108 (1933), pp. 378–410.