ON THE CONVERGENCE OF FOURIER SERIES

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The purpose of this note is to show that a criterion for the convergence of Fourier series, given by Tonelli,* is contained in the Lebesgue criterion.

The conditions of the Tonelli criterion are†

$$\phi(t) = o(1) \text{ as } t \to 0; \tag{1}$$

(2) $\phi(t)$ is absolutely continuous in the interval (ϵ, π), $\epsilon > 0$, or

$$\phi(t) = \int_{t_1}^{t} \phi'(\tau) d\tau, \text{ for } t > 0;$$

(3) meas. $\underline{\lim}_{t \to 0}\, t\phi'(t) \geqq 0$.

Here by meas. $\underline{\lim}$ we mean that in calculating the limit we are permitted to leave out of consideration sets of measure zero. We have to show that, if the above conditions are satisfied, the conditions of the Lebesgue criterion,

$$\Phi(t) = \int_0^t |\phi(\tau)| d\tau = o(t), \tag{4}$$

$$\int_\delta^a |\phi(t+\delta) - \phi(t)| \frac{dt}{t} = o(1), \text{ as } \delta \to 0, \delta > 0, \tag{5}$$

are also satisfied.

Condition (4) is an obvious consequence of (1). Let $\phi'(t) = \phi_1'(t) + \phi_2'(t)$, where

$$\phi_1'(t) = \begin{cases} \phi'(t), & t \subset E \equiv E(\phi'(t) \geqq 0), \\ 0, & t \subset C(E), \end{cases} \qquad \phi_2'(t) = \begin{cases} 0, & t \subset E, \\ \phi'(t), & t \subset C(E). \end{cases}$$

Then $|\phi'(t)| = \phi_1'(t) - \phi_2'(t) = \phi'(t) - 2\phi_2'(t)$. Since $\phi_2'(t) < 0$, condition (3) implies that we have meas. $\underline{\lim}_{t \to 0}\, t\phi_2'(t) = 0$, or $\phi_2'(t) = o(1/t)$, except for a set of measure zero. Since we know that $\delta > 0$ and $\phi(t) = \int_{t_1}^{t} \phi'(\tau) d\tau$ if $t > 0$,

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* L. Tonelli, *Serie Trigonometriche*, 1928, p. 291.

† $\phi(t)$ is defined in the usual way by $\phi(t) = f(x+t) + f(x-t) - 2f(x)$.