# ON EULER'S TOTIENT FUNCTION

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In this note we discuss the equation

$$(1) \qquad k\phi(n) = n - 1,$$

where $k$ is an integer, and $\phi(n)$ is Euler's totient function, giving the number of integers $<n$ and prime to $n$. Our main purpose is to show that if $n$ is a solution of (1), then $n$ is a prime or the product of seven or more distinct primes. One is tempted to believe the stronger statement that (1) has no composite solutions or, in other words, the integer $n$ is a prime if (and only if) $\phi(n)$ divides $n-1$. We have not been able to establish this, however. The proof of the nonexistence of composite solutions of (1) seems about as remote as the proof of the nonexistence of odd perfect numbers and the two problems though not equivalent are not dissimilar.

Let $n$ be a composite solution of (1) and let $a$ be any number prime to $n$; then

$$a^{n-1} = (a^{\phi(n)})^k \equiv 1 \pmod{n},$$

so that $n$ furnishes an example of the failure of the strict converse of Fermat's theorem for all values of $a$ prime to $n$. This involves no contradiction, however. In fact $a^{560} \equiv 1 \pmod{561}$, for all $a$'s prime to 561, although $561 = 3\cdot 11\cdot 17$.

Together with (1) we shall consider the equation

$$(2) \qquad k\phi(n) = n + 1,$$

and show that it has exactly eight solutions if $n$ has less than seven distinct prime factors. The case $k=1$ may be dispensed with since (2) has no solutions and (1) has a solution $n$, if and only if $n$ is a prime. We first give a number of necessary conditions which any solution $n$ of (1) or (2) must satisfy.

THEOREM 1. *If $n>2$, then $n$ is a product of distinct odd primes.*

PROOF. From equations (1) and (2) it is obvious that $n$ must be prime to $\phi(n)$, and since $\phi(n)$ is even for $n>2$, $n$ must be odd.

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