THE USE OF FRACTIONAL INTEGRATION AND DIFFERENTIATION FOR OBTAINING CERTAIN EXPANSIONS IN TERMS OF BESSEL FUNCTIONS OR OF SINES AND COSINES

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1. *Introduction.* Under certain conditions fractional integration or differentiation of a sine function will lead to a Bessel function and vice versa. Likewise the fractional integration or differentiation of a cosine function will lead to Struve's function. It follows, when the process is legitimate, that a known expansion in sines can be converted to an expansion in Bessel functions, or a known expansion in Bessel functions can be converted to a sine expansion, by the simple expedient of term-by-term fractional integration or differentiation.

2. *Fractional Integration and Differentiation of Sine and Cosine.* Fractional integration* with the Heaviside operator p is given by the equation

$$p^{-\nu}f(x) = \int_0^x \frac{(x-\lambda)^{\nu-1}}{\Gamma(\nu)} f(\lambda)d\lambda, \tag{1}$$

where $\nu > 0$. Fractional differentiation* is given by

$$p^{\nu}f(x) = \frac{d^b}{dx^b}\int_0^x \frac{(x-\lambda)^{c-1}}{\Gamma(c)} f(\lambda)d\lambda, \tag{2}$$

where $\nu > 0$, $0 < c < 1$, b is a positive integer, and $\nu = b - c$. If (1) is applied to $f(x) = x^n$ the result is

$$p^{-\nu}x^n = \frac{\Gamma(n+1)x^{n+\nu}}{\Gamma(n+\nu+1)}, \quad (\nu > 0,\ n > -1). \tag{3}$$

If (2) is applied to x^n,

$$p^{\nu}x^n = \frac{\Gamma(n+1)x^{n-\nu}}{\Gamma(n-\nu+1)}, \quad (\nu > 0, n > -1). \tag{4}$$

* For bibliography on fractional integration see H. T. Davis, *The application of fractional operators to functional equations*, American Journal of Mathematics, vol. 49 (1927), pp. 123–142.