NOTE ON THE DISCRIMINANT MATRIX OF AN ALGEBRA*

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The purpose of this note is to extend MacDuffee's normal basis† to a general linear associative algebra.

Let $\mathfrak{A}$ be a linear associative algebra over an infinite field $\mathfrak{F}$, with the basis $e_1, e_2, \cdots, e_n$, and let the constants of multiplication be denoted by c_{ijk}. Let $T_1=(\tau_{rs})$ be the first discriminant matrix of $\mathfrak{A}$, and let $d_h=\sum_k c_{hkk}$. Then $\tau_{rs}=\tau_{sr}=\sum_h c_{srh}d_h$.

If $\mathfrak{A}$ is nilpotent, $d_i=0$, $(i=1, 2, \cdots, n)$,‡ and $T_1=0$. We now suppose that $\mathfrak{A}$ is non-nilpotent and therefore possesses a principal idempotent element e_1.§ Let $\mathfrak{N}$ be the radical of $\mathfrak{A}$, and $\mathfrak{B}$ be the set of elements x of $\mathfrak{A}$ for which $e_1x=0$. Then $\mathfrak{B}<\mathfrak{N}$.¶ It is easily shown that $\mathfrak{A}=e_1\mathfrak{A}+\mathfrak{B}$, where $e_1\mathfrak{A}$ and $\mathfrak{B}$ are algebras whose intersection is zero. Let $e_1\mathfrak{A}=\mathfrak{L}+\overline{\mathfrak{N}}$, where $\overline{\mathfrak{N}}$ is the radical of $e_1\mathfrak{A}$ and $\mathfrak{L}$ is a linear system supplementary to $\overline{\mathfrak{N}}$ in $e_1\mathfrak{A}$. It is not difficult to show that $\mathfrak{N}=\overline{\mathfrak{N}}+\mathfrak{B}$.|| We may therefore select the basis of $\mathfrak{A}$ as $e_1, e_2, \cdots, e_n$, so that e_1 is the principal idempotent selected above, $e_1, e_2, \cdots, e_\sigma$ is a basis for $\mathfrak{L}$, $e_{\sigma+1}, e_{\sigma+2}, \cdots, e_\rho$ a basis for $\overline{\mathfrak{N}}$, and $e_{\rho+1}, e_{\rho+2}, \cdots, e_n$ a basis for $\mathfrak{B}$. Then $d_i=0, (i>\sigma)$,** and $d_1=\sum_k c_{1kk}=\rho>0$, since if x is in $e_1\mathfrak{A}$, we have $e_1x=x$.

Direct computation shows that if $e_1, e_2, \cdots, e_n$ are subjected to a transformation, $e_i'=\sum_j a_{ij}e_j$, the new d's are given by $d_i'=\sum_j a_{ij}d_j$, $(i=1, 2, \cdots, n)$. Hence if we make the non-singular transformation

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† C. C. MacDuffee, Transactions of this Society, vol. 33, p. 427, proves Theorems 1 and 2 only for algebras with a principal unit. The terminology and notation in this paper are in agreement with that of MacDuffee.

‡ L. E. Dickson, *Algebren und ihre Zahlentheorie*, 1927, p. 108.

§ Dickson, loc. cit., p. 100.

¶ Dickson, loc. cit., p. 100.

|| This relation follows directly from Dickson, loc. cit., p. 100, Theorem 5, or it can be proved independently.

** Dickson, loc. cit., p. 108.