Erreurs et Moindres Carrés. By R. Deltheil. Paris, Gauthier-Villars, 1930. v+161 pp.

This treatise is the second paper of an excellent series of monographs on probability, entitled *Traité du Calcul des Probabilités et de ses Applications*, edited by Émile Borel, with the collaboration of L. Blaringhem, C. V. L. Charlier, R. Deltheil, H. Galbrun, J. Haag, R. Lagrange, F. Perrin, and P. Traynard.

On the theoretical side, Deltheil sketches in a masterly way two of the main routes rigorously leading to the so-called Gaussian probability law (Chapters 2, 4, 6, 7). The first of these is the method of moments, using staircase functions, going back to Tchebychef, Bienaymé, Stieltjes, Castelnuovo, and Liapounoff. Deltheil refers the readers to the monograph written by Borel as the first paper of the *Traité* for further details. The second rigorous method is that of Paul Lévy who uses as a "characteristic function" the Stieltjes integral of $e^{ixt}dF(x)$, where $F(x)$ is the cumulative frequency law. This characteristic function goes back to Cauchy, and still further back to Laplace.* Deltheil's treatment here is a felicitous condensation of Lévy's excellent *Calcul des Probabilités*, to which the reader is referred for details. An interesting discussion of the hypotheses required by the two methods of approach to the Gaussian Law is given (pp. 80–82), in which some later work by Fréchet † is also mentioned, whose hypotheses lead to a different law.

In any comprehensive treatment of least squares, some use or at least mention of Bayes' Theorem seems inevitable, in spite of the difficulties involved in making the treatment rigorous. So long as the a priori probability is under control, no special obstacle arises; and Deltheil's Chapter 1 gives a very satisfactory and instructive treatment for this aspect. Chapter 3 picks up Bayes' Theorem again for use with measurements where, in general, the a priori probabilities are not known and cannot be known. The hypothesis of constant a priori probability is accepted temporarily as the only reasonable assumption in case of total ignorance, but the reader is referred to Chapter 9 for release from this objectionable assumption. In Chapter 5, devoted to the principle of the arithmetic mean, the difficulty reaches its culmination. The original "demonstration" of Gauss is given, and some of the numerous objections raised thereto. Deltheil concludes (p. 63) that the critics of Gauss are justified. Poincaré's attempt is given (pp. 59–63) to salvage something from the wreck. Here, the principle of the arithmetic mean as the most probable value leads to the assumption that the a priori probability $\phi(x)$ is a constant. A rather peculiar constant, indeed, this would be; for the integral of $\phi(x)$ from $-\infty$ to $+\infty$ must be unity. Possibly, this "proof" of Gauss, abandoned by Gauss himself (p. 63), has great "historical value"; since nearly every book on least squares incorporates it—often reluctantly. But, perhaps the space devoted to such obsolete material could be better used. Instead of clinging to the untenable "$\phi(x)=$constant," it is possible to show that under very broad conditions the effect of $\phi(x)$

* E. C. Molina, *The theory of probability: some comments on Laplace's Théorie Analytique*, this Bulletin, vol. 36 (1930), pp. 369–392.

† *Sur l'hypothèse de l'additivité des erreurs partielles*, Bulletin des Sciences Mathématiques, 1928.