NOTE ON A THEOREM OF BÔCHER AND KOEBE*

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1. *Introduction.* In this paper a generalization of the following theorem, discovered independently by Bôcher† and Koebe,‡ is established.

THEOREM 1. *If $u(x, y)$ is continuous with its first partial derivatives in a plane region R, and if, for every circle C contained in R,*

$$\int_C \frac{\partial u}{\partial n} ds = 0,$$

where n is the exterior normal to C, then u is harmonic in R.

The generalization obtained is embodied in Theorem 2.

THEOREM 2. *If $v(x, y)$ is harmonic and positive in R, if $u(x, y)$ is continuous with its first partial derivatives in R, and if*

$$\int_C v \frac{\partial u}{\partial n} ds = \int_C u \frac{\partial v}{\partial n} ds \tag{1}$$

for every circle C contained in R, then u is harmonic in R.

Taking v as the constant one in Theorem 2, Theorem 1 is obtained.

Like Theorem 1,§ Theorem 2 has an analog in space, but,

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† Bôcher, M., *On harmonic functions in two dimensions*, Proceedings of the American Academy of Arts and Sciences, vol. 41 (1906), pp. 577–583.

‡ Koebe, P., *Herleitung der partiellen Differentialgleichung der Potentialfunktion aus der Intergraleigenschaft*, Sitzungsberichte der Berliner Mathematischen Gesellschaft, vol. 5 (1906), pp. 39–42.

§ Koebe, loc. cit. For generalizations of Bôcher's and Koebe's Theorem of another type, see G. C. Evans, *Fundamental points of potential theory*, Rice Institute Pamphlets, vol. 7 (1920), pp. 252–329, espe ially.p. 286, and *Note on a theorem of Bôcher*, American Journal of Mathematics, vol. 50 (1928), pp. 123–126; and G. E. Raynor, *On the integro-differential equation of the Bôcher type in three space*, this Bulletin, vol. 52 (1926), pp. 654–658. Evans, using the notion of the potential function of a gradient vector, shows that the conclusion of Theorem 1 holds with much lighter hypotheses both on u and the character of the curves C.