

LIMITS OF APPROXIMATE SOLUTIONS OF A TORSION PROBLEM*

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1. *Introduction.* In the application of Ritz's† method to the solution of a boundary value problem by the direct process of the calculus of variations, one has the difficulty of not knowing what degree of approximation it provides in a given case, except to say that it gives an upper limit to the solution. Analogous to the Ritz method in which the form of the solution (containing parameters determinable by a minimizing process) is approximated by a complete set of independent functions satisfying the boundary conditions but not the differential equation, Trefftz‡ has presented a method adaptable to the Dirichlet problem, and this provides a lower bound. In this process the approximating functions are chosen to satisfy only the differential equation. By application of both of these approximating methods upper and lower bounds of the true minimum are provided.

In this paper these methods are applied to the torsion problem of a right prism whose cross section is symmetrical and is bounded by arcs of two confocal parabolas.

2. *The Torsion Problem.* The torsion problem is the problem§ of determining a function $\phi(x, y)$ such that on the boundary Γ of the region R , it attains the value

$$(1) \quad \phi = \frac{\tau}{2}(x^2 + y^2) + \text{a constant},$$

and

$$(2) \quad \nabla^2\phi = \phi_{xx} + \phi_{yy} = 0$$

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† W. Ritz, *Journal für Mathematik*, vol. 135 (1909), pp. 1-61; *Collected Works*, 1911, pp. 192-250.

‡ E. Trefftz, *Ein Gegenstück zum Ritzschen Verfahren*, *Proceedings of the 2d International Congress for Applied Mechanics*, Zurich, 1926, p. 131.

§ A. E. H. Love, *The Mathematical Theory of Elasticity*, 4th edition, 1927, p. 312; *Handbuch der Physik*, vol. 6, Springer, 1928, p. 143.