CONVERGENCE CRITERIA FOR CONTINUED FRACTIONS*

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1. *Introduction.* The object of this paper is to present two new criteria for the convergence of the continued fraction

$$F(\alpha, z) = \frac{1}{\alpha_1 z +} \frac{1}{\alpha_2 +} \frac{1}{\alpha_3 z +} \cdots,$$

in which the numbers α_i are real and different from zero.

Necessary and sufficient conditions for convergence have been discovered for the case that $\alpha_i > 0$ by Stieltjes;† and by Hamburger‡ when $\alpha_{2i+1} > 0$. There seem to be no necessary and sufficient conditions known for the general case, although several sufficient conditions have been found. Van Vleck§ showed that if k is the greatest modulus of the limit points of the numbers $1/(\alpha_i \alpha_{i+1})$, then $F(\alpha, z)$ converges, except for isolated points, within the circle $|z| = 1/(4k)$. Inasmuch as k may be infinite while at the same time $\alpha_i > 0$, $\sum \alpha_i$ diverges, it follows from the work of Stieltjes that $F(\alpha, z)$ may converge to an analytic limit even when the circular region $|z| = 1/(4k)$ vanishes. The theorems which I shall give include certain cases of this sort.

2. *Notation.* Let $a_1, a_2, a_3, \cdots ; b_1, b_2, b_3, \cdots$, be two infinite sequences of real non-zero numbers connected by the relations

$$(1) \qquad a_{2i} = b_{2i+1}/(\overset{b}{\delta}_{i-1} \overset{b}{\delta}_i), \qquad a_{2i+1} = b_{2i+2}[\overset{b}{\delta}_i]^2,$$

where

$$\overset{b}{\delta}_i = b_1 + b_3 + \cdots + b_{2i+1}.$$

It is easily seen that if we set

$$\overset{a}{g}_i = a_2 + a_4 + \cdots + a_{2i}, \quad (\overset{a}{g}_0 = 0),$$

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† Annales de la Faculté des Sciences de Toulouse, vol. 8, J, pp. 1–122, and vol. 9, A, pp. 1–47.

‡ Mathematische Annalen, vol. 81, pp. 234–319; vol. 82, pp. 120–187.

§ Transactions of this Society, vol. 2, pp. 476–483.