INVERSE TERNARY CONTINUED FRACTIONS

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In Jacobi's extension of the continued fraction algorithm* we are concerned with three series of numbers given by the recursion formulas

$$A_n = q_nA_{n-1} + p_nA_{n-2} + A_{n-3},$$
$$B_n = q_nB_{n-1} + p_nB_{n-2} + B_{n-3},$$
$$C_n = q_nC_{n-1} + p_nC_{n-2} + C_{n-3},$$

with initial values 1, 0, 0 for A; 0, 1, 0 for B, and 0, 0, 1 for C. We have called this series of numbers (A_n, B_n, C_n) the *convergent sets*, and the series of numbers (p_n, q_n) the *partial quotient sets* of a *ternary continued fraction.*† It is well known that if the partial quotient sets recur periodically the ratios A_n/B_n, A_n/C_n and B_n/C_n approach cubic irrationalities except in certain special cases where they approach quadratic irrationalities‡ or where they approach no limit at all. The cubic irrationalities when they exist are connected by a linear fractional relation with the roots of the *characteristic cubic*

$$\begin{vmatrix} A_{k-2}-\rho & B_{k-2} & C_{k-2} \\ A_{k-1} & B_{k-1}-\rho & C_{k-1} \\ A_k & B_k & C_k-\rho \end{vmatrix} = 0,$$

connected with the *purely periodic* ternary continued fraction $(p_1, q_1; p_2, q_2; \cdots; p_k, q_k)$ formed of those partial quotient pairs that recur. This characteristic cubic we write $\rho^3-M\rho^2+N\rho-1=0$, where $M=A_{k-2}+B_{k-1}+C_k$, and

$$N = A_{k-2}B_{k-1} - A_{k-1}B_{k-2} + B_{k-1}C_k - B_kC_{k-1} + C_kA_{k-2} - A_kC_{k-2}.$$

We shall confine ourselves in what follows to purely periodic

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* Jacobi, Werke, vol. VI, pp. 385–426.

† Proceedings of the National Academy of Sciences, vol. 4 (1918), p. 360.

‡ J. B. Coleman, American Journal of Mathematics, vol. 52, No. 4, October, 1930.