A PROOF OF THE IDENTITY OF THE RIESZ INTEGRAL AND THE LEBESGUE INTEGRAL*

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1. *Introduction.* In Acta Mathematica, volume 42, pages 191–205, Friedrich Riesz developed a theory of integration independent of the theory of measure of point sets except in so far as sets of measure zero were involved. His theory yields an integral which he showed to have many of the properties of the Lebesgue integral and which he showed to exist and to be identical with the Lebesgue integral when this latter integral exists. Riesz' treatment yields a theory of measure which has the essential characteristics of the Lebesgue theory. Riesz does not seem to have proved that his development yields an integral which is identical with the Lebesgue integral† and that his notion of measure is the same as the ordinary notion. The present paper does show the entire identity of the Riesz integral and the Lebesgue integral. The terminology and notation of Riesz' paper are used.

THEOREM. *A necessary and sufficient condition that a bounded function on $X: a \leqq x \leqq b$ be measurable is that there exist a sequence of simple functions‡ which approaches this function almost everywhere on X.*

2. *Proof of Necessity.*§ Let $f(x)$ be bounded and measurable on X and let a method of subdivision of X be chosen in such a way that the length of the longest subdivision approaches zero as the number of these subdivisions becomes infinite. $f(x)$ is Lebesgue integrable on X. Let $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ be the subdivision points at the nth stage and let

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† At the bottom of page 199 of Riesz' paper one finds a statement which indicates that he suspected a close relationship between or even the identity of, the two integrals; but he makes no positive statement, and gives no proof.

‡ Also called *step functions* or *horizontal functions.*

§ This part of the theorem is well known. A proof is included here for completeness.