SOME THEOREMS ON PLANE CURVES

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In applying Abel's theorem to hyperelliptic integrals, we are interested in the intersections of certain curves with a curve C of the type $y^2=f(x)$, where $f(x)$ is a polynomial. The functions used in the following are all polynomials of degree indicated by their subscripts. If $f_n(x)\equiv f_k(x)f_{n-k}(x)$ we may without any loss of generality assume that $n\geqq k\geqq n/2$ and this assumption will be made throughout.

LEMMA. *If C is the curve $y^2=f_n(x)\equiv f_k(x)f_{n-k}(x)$, c_1 the curve $y=f_k(x)$ and c_2 the curve $y=f_{n-k}(x)$, then all the finite points of intersection of c_1 and c_2 are on C, and the curve S whose equation is $y=[f_k(x)+f_{n-k}(x)]/2$ is tangent to C at each of these k points.*

Suppose (α, β) is any one of the k points of intersection of c_1 and c_2; then $\beta=f_k(\alpha)$ and $\beta=f_{n-k}(\alpha)$ and therefore $\beta^2=f_k(\alpha)f_{n-k}(\alpha)=f_n(\alpha)$, that is (α,β) is on C. Obviously S passes through the k points of interesection of c_1 and c_2 and hence meets C in these k points. Eliminating y from the equations of S and C we get

$$\left[\frac{f_k(x)+f_{n-k}(x)}{2}\right]^2-f_k(x)f_{n-k}(x)\equiv\left[\frac{f_k(x)-f_{n-k}(x)}{2}\right]^2=0$$

as the equation giving the abscissas of the $2k$ points of intersection of S and C. Since the left hand side of this equation is a perfect square each abscissa is counted twice, and therefore since, in S, y is a one-valued function of x, S is tangent to C at each of these k points.

As an immediate consequence of this lemma we have the following result.

THEOREM 1. *If C is the curve $y^2=\phi_n(x)$, where $\phi_n(e_i)=0$, $(i=1,\cdots,n)$, and (α,β), $(\beta\neq 0)$, is a point on C, and c_1 is the curve of the form $y=\phi_k(x)$ determined by (α,β) and any k of the points $(e_i, 0)$, and c_2 is the curve of the form $y=\phi_{n-k}(x)$ determined by (α,β) and the remaining $n-k$ of the points $(e_i, 0)$, then c_1 and*