AN EXISTENCE THEOREM FOR CHARACTERISTIC CONSTANTS OF KERNELS OF POSITIVE TYPE*

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We define a kernel $K(s, t)$ to be of positive type with respect to a set of functions $H(s)$ if

$$\int_a^b \int_a^b K(s, t)h(s)h(t)dsdt \geqq 0 \tag{1}$$

for every function $h(s)$ of the set $H(s)$.

Let the real kernel $K(s, t)$ be developable in a series of real normalized orthogonal functions $\phi_i(s)$, so that

$$K(s, t) = \sum_{i,j=1}^{\infty} a_{ij}\phi_i(s)\phi_j(t), \tag{2}$$

where

$$a_{ij} = \int_a^b \int_a^b K(s, t)\phi_i(s)\phi_j(t)dsdt, \tag{3}$$

and not all the a_{ij} are zero. We shall prove the following theorem.

THEOREM 1. *If $K(s, t)$ is of positive type with respect to the set of all functions of the form $c_\alpha\phi_\alpha(s)+c_\beta\phi_\beta(s)$, where the c's are real constants, then no coefficient a_{kk} is negative, and no a_{kk} is zero unless $a_{kj}+a_{jk}=0$ for every j.*

Suppose, for some subscript k, we have $a_{kk}<0$. Let $h(s)=\phi_k(s)$ in (1). Then we have, since the functions $\phi_j(s)$ form a normalized orthogonal set,

$$\begin{aligned} &\int_a^b \int_a^b K(s, t)\phi_k(s)\phi_k(t)dsdt \\ &= \int_a^b \int_a^b \sum_{i,j} a_{ij}\phi_i(s)\phi_j(t)\phi_k(s)\phi_k(t)dsdt \\ &= \int_a^b \sum_i a_{ik}\phi_i(s)\phi_k(s)ds = a_{kk} < 0, \end{aligned} \tag{4}$$

and we see that condition (1) is not satisfied.

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