ON THE WEDDERBURN NORM CONDITION FOR CYCLIC ALGEBRAS*

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1. *Introduction.* Let F be any non-modular field, i a root of a cyclic equation in F of degree n and with roots $\theta^r(i)$. Suppose that A is a cyclic algebra with basis

$$i^r y^s, \qquad (r, s = 0, 1, \cdots, n-1),$$

where

$$y^r i = \theta^r(i) y^r, \; y^n = \gamma \text{ in } F.$$

J. H. M. Wedderburn has proved† that A is a division algebra if γ^r is not the norm, $N(a)$, of any a in $F(i)$ for every positive integer r less than n. It has never been shown, however, that this condition is a necessary one; but the problem of finding complete necessary and sufficient conditions has been reduced to the case n a power of a single prime.‡

In the present paper cyclic algebras of order sixteen with the corresponding cyclic quartic in its canonical form§

$$\phi(\omega) \equiv \omega^4 + 2\nu(1+\Delta^2)\omega^2 + \nu^2\Delta^2(1+\Delta^2) = 0$$

such that ν and Δ are in F, and $\tau = 1+\Delta^2$ is not the square of any quantity of F, are considered. The norm $N(a)$ of a polynomial in i is a rather complicated quartic form in four variables, yet we can secure the result that $\gamma^2 = N(a)$ if and only if $\gamma = \alpha^2 - \beta^2\tau$ for α and β in F, a curious property of cyclic quartic fields. When the above equation is satisfied the algebra A is expressible as a direct product of two generalized quaternion algebras. Necessary and sufficient conditions are secured that our algebras A of order sixteen be division algebras, and it is shown that for the particularly interesting case where F is *the field of all rational numbers* the Wedderburn condition is *necessary as well as sufficient.*

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† Transactions of this Society, vol. 15 (1914), pp. 162–166.

‡ See a paper by the author, *On direct products, cyclic algebras, and pure Riemann matrices,* to appear in the Transactions of this Society, January, 1931.

§ See R. Garver, *Quartic equations with certain groups,* Annals of Mathematics, vol. 29 (1928), pp. 47–51.