FUNCTIONAL EQUATIONS FOR TOTIENTS*

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1. *Totient Functions.* Let $p_1, \cdots, p_a$ be the distinct prime factors of n. The number of different sets of k equal or distinct positive integers less than or equal to n whose G. C. D. is prime to n is the Jordan totient $\phi_k(n)$ of order k, and

$$\phi_k(n) \equiv n^k(1 - p_1^{-k}) \cdots (1 - p_a^{-k}). \tag{1}$$

If k is complex, $\phi_k(n)$ is defined by (1). The special case $k=1$ gives Euler's $\phi(n)$. The case $k=0$ is trivial and will be ignored.

We say that the numerical function $f\equiv f(n)$ is *factorable* if $f(1)=1$ (which is adjoined to the definition of $f(n)$ if $f(n)$ is defined arithmetically for $n>1$ but not for $n=1$), and if $f(mn) = f(m)f(n)$ for all pairs of coprime integers m, n.

Factorability is distinct from separability, which we define as follows. Let α denote a variable integer >0 and p a fixed but arbitrary prime. Write $p^\alpha \equiv y$, $p\equiv x$, and regard x, y as independent variables. Then the factorable numerical function f is *separable* if there exist numerical functions g, h such that $f(p^\alpha)\equiv g(x)\,h(y)$. For example, ϕ_k is separable; σ_k, where $\sigma_k(n)$ is the sum of the kth powers of all the divisors of n, is not.

If f is separable, say $f(p^\alpha)\equiv g(x)\,h(y)$, and if further $h(y)$ is of the form $\sum_{j=1}^{s}(a_jy^j+b_jy^{-j})$, where the a_j, b_j are constants and $a_sb_s\neq 0$, we say that f is *simply* separable of *extent* s. Thus ϕ_k is simply separable of extent 1. *Regarding* $f(p^\alpha)$ *as a function of* x *and* y, *we shall write* $f(p^\alpha)\equiv f(x, y)$.

It is well known that ϕ_k is the unique solution f of the functional equation

$$\sum_{d|n} f(d) = n^k, \qquad (n = 1, 2, \cdots). \tag{2}$$

Although a proof of this will not be required, we give one to contrast the algebra involved with another, which will be used.

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