HILBERT AND ACKERMANN ON MATHEMATICAL LOGIC

Grundzüge der theoretischen Logik. By D. Hilbert and W. Ackermann. Berlin, Julius Springer, 1928. 120 pp.

This book deals with mathematical logic very much after the fashion of the first volume of *Principia Mathematica.* The authors begin with a treatment of the theory of elementary functions of propositions, and go on, in their second chapter, to a consideration of the logic of classes and its application to the traditional syllogism; they then take up a discussion of general propositions that involve a use of the notions "some" and "all" as applied to variables denoting individuals; and finally, in their last chapter, they consider propositions involving generalization with respect to functions, in connection with which a discussion of the paradoxes, the theory of types, and the axiom of reducibility, is entailed. These are, of course, all well known topics; but there are certain features of the book that are peculiar to it, and it is to some of these features that we shall direct attention.

The authors begin their formal analysis of elementary functions (p. 22) with the four primitive propositions:

[a] $(p):p \vee p \cdot \supset \cdot p,$

[b] $(p, q):p \cdot \supset \cdot p \vee q,$

[c] $(p, q):p \vee q \cdot \supset \cdot q \vee p,$

[d] $(p, q, r):\cdot p \supset q \cdot \supset :r \vee p \cdot \supset \cdot r \vee q,$

which are identical with four of the five propositions employed in *Principia Mathematica.* They have also a rule of substitution and a rule of inference; but they omit one proposition used by Whitehead and Russell, namely $(p, q, r):\cdot p \cdot \vee \cdot q \vee r : \supset : q \cdot \vee \cdot p \vee r$, because it can be shown to be a logical consequence of the remaining four. Now, a peculiarity of the way in which Hilbert and Ackermann deal with propositions [a]–[d] is this: they endeavor to show (pp. 29 ff.) that these propositions are, at once, *consistent*, *independent*, and *complete*, in the technical senses which these terms bear in connection with ordinary deductive systems. Their arguments in each of these cases call for comment, and we shall consider them in order.

In dealing with the question of consistency, the authors use an interpretational method, involving arithmetical products of 0 and 1. Of course, expressions [a]–[d] express propositions, and thus do not admit of interpretation as they stand; so that we must first abstract from the particular meanings of the symbols in question, and then re-interpret these symbols arithmetically. We need not concern ourselves here with the details of this argument; it involves showing that expressions [a]–[d], when given the arithmetical interpretation in question, are such that they, together with all the arithmetical products that can be derived from them by means of the two rules of deduction, have the value 0; a one-to-one correspondence is assumed to hold between these arithmetical propositions and the symbols, and again between the symbols and the original logical propositions; we are then told that if the original propositions could lead to a contradiction, some arithmetical product derivable from the primitive expressions would have to have the value 1.