MENGER ON THEORY OF DIMENSIONS

Dimensionstheorie. By Karl Menger. Leipzig and Berlin, B. G. Teubner, 1928. iv+319 pp.

There is an important phase in the development of modern point set theoretical geometry which has been closely associated with the concept of dimensionality,—we refer to the attempt to create precise mathematical meaning for the simple geometric spaces of our intuition in terms of primitive non-arithmetical concepts. That the idea of dimensionality should have come into play and itself have been studied and made precise is indeed natural, since the curves, surfaces, and solids of our experience furnish the very basis for our intuitive ideas of dimensionality. The simple arithmetic definition of dimensionality, however, as the number of parameters required to define a space, while useful in ordinary geometry, was of course entirely inadequate when the more abstract spaces came into consideration, and it became highly desirable that dimensionality be relieved of its arithmetical vesture and be based on the inner structure of space itself.

Consider the spaces of our experience. What non-arithmetic relations among them are intuitively certain? The following immediately suggest themselves: a solid can be separated into several parts by one or more surfaces, a surface by curves and a curve by points. It was Poincaré who in 1912 suggested that precisely this type of phenomenon might lead to a satisfactory non-arithmetic definition of dimensionality,—a definition by recurrence. A space may be called n-dimensional, he suggested, if it can be separated into several parts by means of continua of $n-1$ dimensions.

Although the Poincaré definition was far from satisfactory either in precision or in content, it must be regarded as of the highest historical importance, since it indicated the possibilities of definition by recurrence, and was moreover essentially topological. These virtues were recognized by Brouwer, who in 1913 slightly modified the content of the definition and stated it in terms of the topologically precise notions of separation and connectedness; as a basis for recurrence, a zero-dimensional space was defined to be one which contained no continuum as subset. Brouwer showed that this "natural" definition of dimensionality satisfied the formal requirement of yielding the number n when applied to a cartesian S_n.

The dimensionality definition has since undergone further modification. If dimensionality was to be studied per se, it was of course desirable to arrive at a definition which would give rise to a theory of the highest generality and simplicity. To this end the basis for recurrence was altered, and what was of more significance, the concept of dimensionality as a local property was introduced. In its modern form the definition is as follows: a space is at most n-dimensional if each point is contained in an arbitrarily small neighborhood with a boundary of dimensionality at most $n-1$; (-1)-dimensional spaces are null spaces. A space which fails to be at most $(n-1)$-dimensional is at least n-dimensional. A space is n-dimensional if it