THE EXISTENCE OF THE LEBESGUE-STIELTJES INTEGRAL*

BY R. L. JEFFERY

A definition of a Lebesgue-Stieltjes integral of a function $f(x)$ defined on (a, b) with respect to a non-decreasing function $V(x)$ bounded on (a, b) has been given by Hildebrandt.†

This definition involves the idea of the measurability of f with respect to V. If α is the interval $a'<x<b'$, then $V(\alpha)=V(b'-0)-V(a'+0)$. Let a set E be enclosed in a finite or countably infinite set of non-overlapping open intervals $A\equiv\alpha_1, \alpha_2, \cdots$. Let $V(E)$ be the lower limit of $V(A)=\sum V(\alpha_i)$ for all possible enclosures A. In the same way define $V(CE)$. When

$$V(E)+V(CE)=V(a,b)=V(b)-V(a), \tag{1}$$

the set E is said to be measurable relative to V. If for all real values of l the set for which $f>l$ satisfies (1), then f is measurable relative to V. Hobson‡ gives a definition which involves a different formulation of the same idea. To state this we shall make use of the following correspondence between the points of $\alpha=V(a)\leqq u\leqq V(b)=\beta$ and $a\leqq x\leqq b$. First, if x is a point of discontinuity of V, then x goes by means of $u=V(x)$ into the closed interval $V(x-0)\leqq u \leqq V(x+0)$. There will then correspond to each u on (α, β) at least one value of x on (a, b). If to a value of u there corresponds more than one value of x, then V is constant throughout an interval, and x_u shall be the lower end point of this interval, or the lower bound of points of the interval in case it is open. If $f(x)$ is any function defined on (a, b), then $\psi(u)$ is defined by $\psi(u)=f(x_u)$ and

$$LS\int_a^b f(x)dV(x) = L\int_\alpha^\beta \psi(u)du,$$

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† This Bulletin, vol. 24, pp. 188–190.

‡ *Theory of Functions of a Real Variable*, 3d ed., vol. I, §445.