THREE THEOREMS ON CLOSURE OF BIORTHOGONAL SYSTEMS OF FUNCTIONS*

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1. *Introduction.* Professor Birkhoff has given a theorem† which asserts the closure of a given normal orthogonal set of functions provided it is sufficiently near to another such set which is closed. In its applications this theorem is frequently used in conjunction with asymptotic forms for the functions involved, and the information at hand applies only to a portion of the entire set. For this reason an extension of the theorem to sets which involve a certain lack of closure is useful. Such an extension together with a generalization of the theorem to biorthogonal systems of functions is given below in Theorem I.

In Theorems II and III the discussion is restricted to the closure of such biorthogonal systems as are composed of the characteristic functions of an integral equation.

A set of continuous functions $\{u_n(x)\}$ will be said to involve a k-fold lack of closure on a given interval if there exist precisely k continuous functions not identically zero which are linearly independent and are orthogonal on the interval to every function of the set.

THEOREM I. *Let* $\{u_n(x), v_n(x)\}$, *that is,*

$$\left\{\begin{array}{l} u_0(x),\ u_1(x),\ u_2(x),\ \cdots, \\ v_0(x),\ v_1(x),\ v_2(x),\ \cdots, \end{array}\right. \tag{1}$$

be a normalized biorthogonal system of continuous functio on the interval $a \leqq x \leqq b$, *and let* $\{\bar{u}_n(x), \bar{v}_n(x)\}$, *i. e.,*

$$\left\{\begin{array}{l} \bar{u}_0(x),\ \bar{u}_1(x),\ \bar{u}_2(x),\ \cdots, \\ \bar{v}_0(x),\ \bar{v}_1(x),\ \bar{v}_2(x),\ \cdots, \end{array}\right. \tag{2}$$

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† Proceedings of the National Academy, vol. 3 (1917), pp. 656–659.