ON SMALL DEFORMATIONS OF CURVES

BY C. E. WEATHERBURN

1. *Introduction.* This paper is concerned with small deformations of a single tortuous curve, of a family of curves on a given surface, and of a congruence of curves in space. In all cases, the displacement s is supposed to be a small quantity of the first order, quantities of higher order being negligible.*

2. *Single Twisted Curve.* Consider first a given curve in space. The position vector $\boldsymbol{r}$ of a point on the curve may be regarded as a function of the arc-length s of the curve, measured from a fixed point on it. Let $\boldsymbol{t}$, $\boldsymbol{n}$, $\boldsymbol{b}$ be the unit tangent, principal normal and binormal. These are connected with the curvature κ and the torsion τ as in the Serret-Frenet formulas. Imagine a small deformation of the curve, such that the point of the curve originally at $\boldsymbol{r}$ suffers a small displacement s, its new position vector $\boldsymbol{r}_1$ being then

$$\boldsymbol{r}_1 = \boldsymbol{r} + \mathsf{s}. \tag{1}$$

Let a suffix unity be used to distinguish quantities belonging to the deformed curve, and let primes denote differentiations with respect to the arc-length s. Then the element $d\boldsymbol{r}_1$ of the deformed curve, corresponding to the element $d\boldsymbol{r}$ of the original, is given by $d\boldsymbol{r}_1 = d\boldsymbol{r} + d\mathsf{s}$, and its length ds_1 by

$$(ds_1)^2 = (d\boldsymbol{r}_1)^2 = (d\boldsymbol{r})^2 + 2d\boldsymbol{r}\cdot d\mathsf{s} = ds^2(1 + 2\boldsymbol{t}\cdot\mathsf{s}').$$

Consequently $ds_1 = ds(1 + \boldsymbol{t}\cdot\mathsf{s}')$.

The quantity $\boldsymbol{t}\cdot\mathsf{s}'$ represents the increase of length per unit length of the curve, or the extension of the curve at

*See also a paper by Perna, Giornale di Matematiche, vol. 36 (1898), pp. 286–299; and another by Salkowski, Mathematische Annalen, vol. 66 (1908), pp. 517-557.