# A THEOREM CONCERNING DIRECT PRODUCTS*

BY LOUIS WEISNER†

The theorem in question may be stated as follows. *A group of order mn, where m and n are relatively prime, in which every element whose order divides m is commutative with every element whose order divides n, is the direct product of two groups of orders m and n.*

Burnside‡ has proved the theorem for the special case in which either $m$ or $n$ is a power of a prime. Hence, to prove the theorem, we need only show that it is true for groups of order $mn$ if it is true for groups of order $<mn$. To avoid trivial cases we assume that $m$ and $n>1$. We denote by $m_1$, $m_2$, and $n_1$, $n_2$, divisors $>1$ of $m$ and $n$ respectively.

If the group $G$ contains an element of order $m$, let $p$ be a prime factor of $m$ and $p^\alpha$ the highest power of $p$ that divides $m$. Every element of $G$ whose order divides $p^\alpha$ is commutative with every element whose order is prime to $p$. It follows from the special case referred to, that $G$ contains an invariant subgroup of order $mn/p^\alpha$, which contains an invariant subgroup of order $n$, as every element whose order divides $m/p^\alpha$ is commutative with every element whose order divides $n$.

We suppose now that $G$ contains no element of order $m$. The normaliser $H$ of an element $s$ of order $m_1$ includes all the Sylow subgroups of $G$ whose orders divide $n$, and is therefore of order $m_2n$, where $m_2 \geqq m_1$. If $m_2<m$, $H$ contains an invariant subgroup of order $n$. If $m_2=m$, $s$ is invariant under $G$. An element $t$ of $G$ corresponding to an element $t'$ of $G/(s)$ of order $n_1$ is of order $n_1\mu$, where $\mu$ divides $m_1$. Hence $G$ contains§ two elements $t_1$ and $s_1$ of orders $n_1$ and $\mu$ respectively, such that

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† National Research Fellow.

‡ W. Burnside, *Theory of Groups*, 2d edition, p. 327.

§ Loc. cit., p. 16.