ON CONTINUITY IN SEVERAL VARIABLES*

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The following theorem on the continuity of a function of several variables is contained implicitly in a theorem on the existence of solutions of differential equations by Carathéodory.† It is of a general nature and independent of the context in which it is found. It is, therefore, worth while isolating and signalizing it.

THEOREM. *Hypotheses*: (1) $f(x, [y])$ *is defined in the domain* $R: a<x<b,\ y_0-k<[y]<y_0+k$, *where* $[y]\equiv(y_1, \cdots, y_n)$.

(2) *For each* $[y]$ *in* R, $f_x'(x, [y])$ *exists almost everywhere on* $a<x<b$, *and is summable on* $a<x<b$.

(3) $|f_x'(x, [y])|\leqq M(x)$, *where* $M(x)$ *is summable on* $a<x<b$.

(4) *For every* x *on* $a<x<b$, $f(x, [y])$ *is continuous in* $[y]$ *at* $y_1=y_0$.

Conclusion: $f(x, [y])$ *is continuous in* $(x, [y])$ *at* $(x, [y_0])$ *where* x *is on* $a<x<b$.

PROOF. Let us put $F_1\equiv f(x+h,[y_0+k_i])$, where $|k_i|\leqq k$; $F_2\equiv f(x,[y_0+k_i])$; $F_3\equiv f(x,[y_0])$. From hypothesis (2), we have

$$F_1 - F_2 = \int_x^{x+h} f_t'(t, [y_0 + k_i])dt,$$

which becomes by hypothesis (3),

$$|F_1 - F_2| \leqq \int_x^{x+h} M(t)dt. \tag{1}$$

Also, from hypothesis (4), it follows that given a positive ϵ, there exists a positive number k_{xy_0}', such that

$$|F_2 - F_3| < \epsilon, \quad \text{for} \quad |k_i| < k_{xy_0}'. \tag{2}$$

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† *Vorlesungen ueber Reelle Funktionen*, Leipzig, 1918, p. 678, Satz 5.