A RAY OF NUMERICAL FUNCTIONS OF r ARGUMENTS*

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1. *Introduction.* Starting with any set Σ of elements we may combine them according to rules having the formal properties (commutativity, associativity, distributivity) of algebraic addition (A), multiplication (M), subtraction (S), and division (D), provided the resulting combinations can be assigned self-consistent interpretations. Assuming this to be the case for a given Σ, we may then investigate the properties of systems Σ' closed under one or more of A, M, D, S. It is not necessary to consider the properties of Σ' with respect to those of A, M, D, S that are omitted. There are thus conceivable precisely 15 systems Σ'. So far as systems Σ' consisting of numbers (rational integers, rational numbers, algebraic numbers, algebraic integers) are concerned, it appears that 4 of the possible 15 have been deemed of sufficient interest to receive technical names. These are as follows: AS, module (Modul); $AMDS$, field (Körper); AMS, ring (Ring); MD, ray (Strahl), the last being due to Fueter.†

The elements of Σ' need not be numbers to ensure interesting results, for example the algebra of classes and that of the relative product. Further significant theories have evolved from mere ova of Σ''s; thus the theory of partitions is the Σ' generated by a sort of parthenogenesis by A alone from given rational integers. Doubtless with the continued evolution of arithmetic the neglected 11 will also be born, baptized, and investigated. Should this indeed come to pass it is fortunate that algebra has but 4, not 4000, fundamental operations.

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† Introduced in his Dissertation; now current. Cf. *Synthetische Zahlentheorie* (Göschen's Lehrbücherei, 1921).