BOLZANO ON PARADOXES

Paradoxien des Unendlichen. By Bernard Bolzano, mit Anmerkungen versehen von Hans Hahn. Leipzig, Meiner, 1921. 12 + 157 pp.

This is a new edition, with notes by Hans Hahn, of the book first published in 1851, three years after Bolzano's death. It appears to have remained for many years almost unknown; for L. Couturat makes no mention of it in his book *De L'Infini Mathématique* (1896) except to state in a note that he came across it when his own book had already been entirely printed. Stolz includes a consideration of this book in an article (MATHEMATISCHE ANNALEN, 1881, p. 255) giving an estimate of Bolzano's work and influence, and states that several years before Cauchy published his lectures on the calculus, Bolzano had developed the fundamental concepts of the calculus which in many respects agree with those of Cauchy, but which in important respects are more complete. Furthermore Hankel attributes to Bolzano priority over Cauchy of the proper conception of infinite series.

The reviewer does not think it fair to criticize severely from the standpoint of present standards of rigor a book written about seventy-five years ago; but in view of the high estimates put on Bolzano's work, it does seem desirable to mention a few of the errors.

Bolzano is concerned with a discussion of Gergonne's solution of the series

$$a - a + a - a + \cdots. \tag{1}$$

The solution is as follows. Let x be the value of the series; then

$$x = a - a + a - a + \cdots = a - (a - a + a - a + \cdots),$$

that is,

$$x = a - x, \qquad \text{and} \qquad x = \frac{a}{2}.$$

Bolzano criticizes this solution in two ways. In the first place he says that the series in parentheses is not identical with the series (1), because regarded as a set of terms it lacks the first term a. This assertion of Bolzano's is of course erroneous; in fact if it be granted that the series (1) have a value at all, then Gergonne's solution is correct. In the second place Bolzano objects altogether to attaching a value to the series (1). While this objection is quite legitimate, the *grounds* for the objection are not. The series can have no value, Bolzano says, since if it did have, it would simultaneously be equal to 0, a and $-a$, inasmuch as it can be written in several forms, as follows:

$$\begin{aligned} a - a + a - a + \cdots & \\ &= (a - a) + (a - a) + \cdots \\ &= a + (-a + a) + (-a + a) + \cdots \\ &= (-a + a) + (-a + a) + \cdots \\ &= -a + (a - a) + (a - a) + \cdots. \end{aligned}$$