ON THE RIEMANN ZETA FUNCTION *

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In this note a functional relation for the Riemann zeta function is established, and its relation to the well known formula given by Riemann is pointed out. It is believed that the relation is new.

Recall that the even elliptic theta constants are

$$(1)\qquad \begin{cases} \vartheta_{00}(x) = 1 + 2\sum_{n=1}^{\infty} q^{4n^2} = 1 + 2\phi_{00}(x), \\ \vartheta_{01}(x) = 2\sum_{n=1}^{\infty} q^{(2n-1)^2} = 2\phi_{01}(x), \\ \vartheta_{10}(x) = 1 - 2\sum_{n=1}^{\infty}(-1)^{n-1} q^{4n^2} = 1 - 2\phi_{10}(x), \end{cases}$$

with $q = e^{-(\pi/4)x^2}$. These series converge uniformly and absolutely for $0 < \delta \leqq x$. Furthermore

$$\vartheta_{00}\left(\frac{1}{x}\right) = x\vartheta_{00}(x), \qquad \vartheta_{01}\left(\frac{1}{x}\right) = x\vartheta_{10}(x), \qquad \vartheta_{10}\left(\frac{1}{x}\right) = x\vartheta_{01}(x).$$

Let

$$(2)\qquad \vartheta(x) = x^{1/2}[\vartheta_{01}(x) + \vartheta_{10}(x) - \vartheta_{00}(x)];$$

then

$$(3)\qquad \vartheta\left(\frac{1}{x}\right) = \vartheta(x).$$

Also

$$1 + 2\phi_{00}\left(\frac{1}{x}\right) = x[1 + 2\phi_{00}(x)]$$

with two similar relations for ϕ_{01} and ϕ_{10}. From these follow

$$(4)\qquad \begin{cases} \lim_{x\to 0} x\phi_{00}(x) = \frac{1}{2}, & \lim_{x\to 0} x\phi_{10}(x) = 0, \\ \lim_{x\to 0} x\phi_{01}(x) = \frac{1}{2}, & \lim_{x\to 0} x^h\,\vartheta(x) = 0, \end{cases}$$

where h is any constant. To prove the last of the expressions

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