CONVEX DISTRIBUTION OF THE ZEROS OF STURM-LIOUVILLE FUNCTIONS*

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Introduction. In an investigation concerning the distribution in the complex plane of the zeros of the functions of the parabolic cylinder the author of the present note was confronted with the following problem. Given a linear differential equation of the second order of the type

$$w'' + G(z)w = 0, \tag{1}$$

where $G(z)$ is analytic in the region under consideration, *how does the nature of the argument of $G(z)$ affect the distribution of the zeros of a given solution $w(z)$ of* (1)?

We have some results which throw light on this general question. Roughly speaking, one may say that the argument of $G(z)$ affects the *orientation* of the zeros in the complex plane, whereas the absolute value of $G(z)$ seems to affect the *density* of the distribution. As the problem and the results obtained seem to have some intrinsic value aside from their bearing on the special problem, mentioned above, their separate publication may perhaps be justified.

1. *Bounded Argument of $G(z)$.* Let us consider a convex region B in the z-plane in which $G(z)$ is single-valued and analytic and in which, furthermore, the argument of $G(z)$ is restricted as follows:

$$\vartheta_1 \geqq \arg G(z) \geqq \vartheta_2 \quad (\pi > \vartheta_1 - \vartheta_2 \geqq 0). \tag{2}$$

Let us construct a set of polygonal lines (p) in B in the following manner. Take any point P_0 in the interior of B. Draw from this point all rays which form an angle θ_{P_0} with the positive real axis such that

$$-\tfrac{1}{2}\vartheta_1 < \theta_{P_0} < -\tfrac{1}{2}\vartheta_2. \tag{3}$$

Take a second point P_1 in B on any of these rays and use it as vertex for a second set of rays with the same limitation on the slope-angle θ_{P_1} as on θ_{P_0}. Then choose a third point P_2 and so on. The polygonal line so obtained forms an ele-

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