EXTENSIONS OF DIRICHLET MULTIPLICATION AND DEDEKIND INVERSION*

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1. *Introduction.* With Landau,† let us define *Dirichlet multiplication* to be the following formal process of combining two sequences α_n, β_n $(n = 1, 2, \cdots)$ to form a third sequence γ_n, $\gamma_n = \Sigma\alpha_l\beta_m$, the sum extending to all integers l, $m > 0$ such that $lm = n$. Henceforth, unless otherwise stated, we assume the four rational algebraic operations to be purely formal.‡ When any of these operations in relation to a given system of elements have special interpretations, examples of which are noted in a moment, they will be called *specific.* Restating Landau's definition, let us denote by α_n, β_n $(n = 1, 2, \cdots)$ two classes of elements which are such that the product of any element of one class by an element of the other has a unique significance, and likewise for any sum of such formal products. Then the *Dirichlet product* of these classes is the class $(\alpha, \beta)_n$ $(n = 1, 2, \cdots)$, where $(\alpha, \beta)_n = \Sigma\alpha_l\beta_m$ $(lm = n)$.

When the α's and β's are known, the process of determining the φ's from the relation $(\alpha, \varphi)_n = \beta_n$ for n an arbitrary integer > 0 will be called *Dedekind inversion.* It will be shown that the extension of this which we have in view includes Dedekind's inversion in the theory of numbers.§

The interpretations of the general theorems in a specific case will depend only upon the meanings assigned to the elements and to the rational operations upon those elements. Thus, if the elements are numbers and the rational operations are as in arithmetic, the interpretation is obvious; if the elements are classes, multiplication and addition are logical,

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† Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, vol. 2, p. 671. We replace his series by sequences.

‡ As in the customary definitions of an abstract field, cf. Dickson, *Linear Groups*, pp. 5–6. When some of the operations in a field are specific, the field will be called special. See also § 6.

§ The inversion is usually attributed to Dedekind, although it was published simultaneously by Liouville. Cf. Dickson's *History*, vol.1, p. 441.