INFINITE SYSTEMS OF FUNCTIONS.

BY PROFESSOR W. E. MILNE.

THE study of infinite systems of functions is approached in this paper from an elementary point of view, and by easy steps there are derived results of considerable generality. It is found that every enumerable system of real functions whose squares are integrable in the sense of Lebesgue either is orthogonal, or possesses an adjoint, or is essentially linearly dependent. Corresponding to every normalized system of functions $\varphi_1, \varphi_2, \varphi_3, \cdots$, is a set of constants $\lambda_1, \lambda_2, \lambda_3, \cdots$, where $1 \geqq \lambda_i \geqq 0$ for every i, with the properties:

A necessary and sufficient condition that the system

(*a*) be orthogonal is that $\lambda_i = 1$ for every i,

(*b*) possess an adjoint is that $\lambda_i > 0$ for every i,

(*c*) be essentially linearly dependent is that $\lambda_i = 0$ for some i.

§ 1.

For simplicity the discussion is limited to real functions of a single real variable.* The symbol Ω denotes the class of all such functions whose squares are integrable in the sense of Lebesgue in the interval (a, b). Functions of Ω, as well as sums and products of such functions are integrable in (a, b). The word "function" in this paper will always mean a function of class Ω, and all properties stated of a system of class Ω will be understood to hold throughout the interval (a, b). It is assumed that the reader is familiar with the terms norm of a function, normalized system, orthogonal system, biorthogonal systems, complete systems, essential linear dependence (of a finite number of functions), convergence in the mean, etc.†

§ 2.

Let there be given an enumerable system $[\varphi]$ of normalized functions $\varphi_1, \varphi_2, \varphi_3, \cdots$, of class Ω. First we investigate the

* The reader will see that the methods may be extended to more general systems.

† Definitions of all these terms are given by Brand "On infinite systems of linear integral equations," *Annals of Math.*, vol. 14 (1913), p. 101.