NOTE ON CONVERGENCE TESTS FOR SERIES AND ON STIELTJES INTEGRATION BY PARTS.

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(Read before the American Mathematical Society September 4, 1919.)

THE obvious identity

$$(1)\qquad [c_k^{(1)}c_k^{(2)}\cdots c_k^{(n)} - a_k^{(1)}a_k^{(2)}\cdots a_k^{(n)}] = [a_k^{(2)}a_k^{(3)}\cdots a_k^{(n)}(c_k^{(1)} - a_k^{(1)})] + \sum_{i=2}^{n-1}[c_k^{(1)}\cdots c_k^{(i-1)}a_k^{(i+1)}\cdots a_k^{(n)}(c_k^{(i)} - a_k^{(i)})] + [c_k^{(1)}\cdots c_k^{(n-1)}(c_k^{(n)} - a_k^{(n)})]$$

may serve in several ways for the investigation of the convergence of series. In each it is to be used as a means of relating the $n+1$ sums of the first m terms of $n+1$ different series the general kth terms of which in order are the $n+1$ bracketed expressions in the foregoing identity. *It is obvious that the convergence of any n of these series implies the convergence of the remaining one.* Moreover, when they all converge, the identity leads to an obvious relation among the $n+1$ sums.

The effectiveness of the whole class of theorems arising thus lies in the fact that the series whose convergence is asserted may be, and in important cases is, conditionally convergent while the auxiliary series are absolutely convergent.

To arrive at a special instance of the theorems thus indicated, let us put $a_k^{(i)} = c_{k-1}^{(i)}$ with $c_0^{(i)} = 0$ and sum in (1) with respect to k from 1 to m. Then we have

$$(2)\qquad \begin{aligned} c_m^{(1)}c_m^{(2)}\cdots c_m^{(n)} &= \sum_{k=1}^{m} c_{k-1}^{(2)}c_{k-1}^{(3)}\cdots c_{k-1}^{(n)}(c_k^{(1)} - c_{k-1}^{(1)}) \\ &+ \sum_{i=2}^{n-1}\sum_{k=1}^{m} c_k^{(1)}\cdots c_k^{(i-1)}c_{k-1}^{(i+1)}\cdots c_{k-1}^{(n)} \\ &\times (c_k^{(i)} - c_{k-1}^{(i)}) + \sum_{k=1}^{m} c_k^{(1)}\cdots c_k^{(n-1)}(c_k^{(n)} - c_{k-1}^{(n)}). \end{aligned}$$

From this relation it is easy to see that we have the following result:*

* The theorems of the paper are given in forms suitable for ready use where applicable. In view of the general remark in the first paragraph it is clear that stronger (though less simply stated) theorems may be obtained from identity (2). A similar remark may be made about other results in the paper.